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Finite-dimensional behaviour for the Benjamin–Bona–Mahony equation

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Abstract. This paper deals with the asymptotic behaviour of solutions for the Benjamin–Bona–Mahony equation. We first show the existence of the global weak attractor for this equation in H^1. And then by an idea of Ball, we prove that the global weak attractor is actually the global strong attractor. The finite-dimensionality of the global attractor is also established.

1. Introduction

In this paper, we investigate the finite-dimensional behaviour of solutions for the Benjamin–Bona–Mahony equation given by

$$u_t - u_{xxt} - \nu u_{xx} + (f(u))_x = g(x) \qquad \text{in } \Omega \times R^+ \tag{1.1}$$

where $\nu > 0$, $f : R \to R$ is a smooth function, $g(x) \in L^2(\Omega)$, $\Omega \subset R$ is a bounded interval.

We note that if $f(u) = u + \frac{1}{2}u^2$, then equation (1.1) becomes

$$u_t - u_{xxt} - \nu u_{xx} + u_x + uu_x = g(x). \tag{1.2}$$

Equation (1.2) has been proposed as a model for propagation of long waves and incorporates nonlinear dispersive and dissipative effects. The existence and uniqueness of solutions for this equation has been investigated by many authors, such as Bona and Dougalis [1], Bona and Smith [2], Showalter [3], Amick *et al* [4]. In the case $\nu = 0$, this equation has been studied by Benjamin *et al* [5], Bona and Bryant [6], Medeiros and Miranda [7], Medeiros and Menzala [8], Albert [9], Biler [10], and the references therein.

The aim of this paper is to derive the existence of the global attractor for equation (1.1) in $H^1(\Omega)$ which has finite Hausdorff and fractal dimensions. Since the nonlinear semigroup $S(t)$ defined by equation (1.1) is not compact in $H^1(\Omega)$, we cannot construct the global attractor by the method introduced by Temam [11], or Constantin *et al* [12]. Here we first employ the techniques developped by Ghidaglia [13] to show the existence of global weak attractor for equation (1.1) in $H^1(\Omega)$. For that purpose, it is necessary that the semigroup $S(t)$ should be weakly continuous in $H^1(\Omega)$ for every $t > 0$. Here we apply a direct method to establish the weak continuity of $S(t)$ in $H^1(\Omega)$. After we obtain the existence of the global weak attractor, by an energy equation and an idea of Ball [15] we conclude that the global weak attractor is actually the global strong attractor for $S(t)$ in $H^1(\Omega)$.

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This paper is organized as follows. In section 2, we recall some facts about the solution semigroup $S(t)$. By a direct method we show that $S(t)$ is weakly continuous in $H^1(\Omega)$ for every $t > 0$. In section 3, we establish the uniform *a priori* estimates in $H^1(\Omega)$. Then by an idea of Ball [15] we show the existence of the global attractor for $S(t)$ in $H^1(\Omega)$.

2. The nonlinear semigroup

We consider the following Benjamin–Bona–Mahony equation:

$$u_t - u_{xxt} - \nu u_{xx} + (f(u))_x = g(x) \qquad (x, t) \in \Omega \times R^+ \tag{2.1}$$

with the initial condition

$$u(x, 0) = u_0(x) \qquad x \in \Omega \tag{2.2}$$

and the periodic boundary condition

$$\Omega = (0, L) \qquad \text{and} \qquad u \text{ is } \Omega\text{-periodic} \tag{2.3}$$

where ν is a positive constant, $f : R \to R$ is a smooth function, $g(x) \in L^2_{\mathrm{per}}(\Omega)$. Similar to the methods used in [1] or [7] we can easily deduce that $\forall u_0 \in H^1_{\mathrm{per}}(\Omega)$, problem (2.1)–(2.3) possesses a unique solution $u(t)$ defined on R^+ such that

$$u(t) \in L^\infty(0, T; H^1(\Omega)) \qquad \frac{\partial u}{\partial t} \in L^\infty(0, T; H^1(\Omega)) \qquad \forall T > 0.$$

This shows that system (2.1)–(2.3) defines a solution semigroup $S(t)$ which maps $H^1(\Omega)$ to $H^1(\Omega)$ such that $S(t)u_0 = u(t)$, the solution of problem (2.1)–(2.3).

Let $H = L^2_{\mathrm{per}}(\Omega)$ be the Hilbert space endowed with its usual inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$, $\|\cdot\|_p$ denote the norm of $L^p(\Omega)$ for all $1 \leqslant p \leqslant \infty (\|\cdot\|_2 = \|\cdot\|)$. $\|\cdot\|_X$ denotes the norm of any Banach space X.

For later purpose, we first establish the following.

Theorem 2.1. Assume that $g(x) \in L^2_{\mathrm{per}}(\Omega)$, $u_0 \in H^1(\Omega)$. Then the dynamical system $S(t) : H^1(\Omega) \to H^1(\Omega)$ is weakly continuous for every $t > 0$.

Proof. $\forall t_1 > 0$ fixed, we shall show $S(t_1)$ is weakly continuous from $H^1(\Omega)$ to $H^1(\Omega)$. Assume now that

$$u_{0k} \to w_0 \qquad \text{weakly in } H^1(\Omega) \tag{2.4}$$

our aim is to show that $S(t_1)u_{0k} \to S(t_1)w_0$ weakly in $H^1(\Omega)$. Choose $T > t_1$, and denote by $u_k(t) = S(t)u_{0k}$, $w(t) = S(t)w_0$. Since the weak convergence implies the boundedness, it follows that

$$\|u_{0k}\|_{H^1(\Omega)} \leqslant R \tag{2.5}$$

where R is a constant independent of k.

Note that $u_k(t)$ satisfies

$$u_{kt} - u_{kxxt} - \nu u_{kxx} + (f(u_k))_x = g(x). \tag{2.6}$$

Taking the inner product of (2.6) with u_k in H, we find that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_k\|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_{kx}\|^2 + \nu\|u_{kx}\|^2 + \int_\Omega (f(u_k))_x u_k \,\mathrm{d}x = (g, u_k). \tag{2.7}$$

Set

$$F(s) = \int_0^s f(\tau)\,\mathrm{d}\tau$$

then we have

$$\int_\Omega (f(u_k))_x u_k \,\mathrm{d}x = -\int_\Omega f(u_k)u_{kx} = -\int_\Omega \frac{\partial}{\partial x}F(u_k)\,\mathrm{d}x = 0. \tag{2.8}$$

It comes from (2.7), (2.8) that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|u_k\|^2 + \|u_{kx}\|^2) + 2\nu\|u_{kx}\|^2 \leqslant \|g\|^2 + \|u_k\|^2. \tag{2.9}$$

By (2.9) and the Gronwall lemma we see that

$$\begin{aligned}\|u_k(t)\|^2 + \|u_{kx}(t)\|^2 &\leqslant \mathrm{e}^t(\|u_k(0)\|^2 + \|u_{kx}(0)\|^2) + \|g\|^2\mathrm{e}^t \\ &\leqslant R^2\mathrm{e}^t + \|g\|^2\mathrm{e}^t \qquad \text{(by (2.5))}.\end{aligned}$$

And hence

$$\|u_k(t)\|^2_{H^1(\Omega)} \leqslant C \qquad \forall 0 \leqslant t \leqslant T \tag{2.10}$$

where C is a constant depending on T.

Taking the inner product of (2.6) with u_{kt} in H, we find that

$$\begin{aligned}\|u_{kt}\|^2 + \|u_{kxt}\|^2 &= -\nu(u_{kx}, u_{kxt}) + (f(u_k), u_{kxt}) + (g, u_{kt}) \\ &\leqslant \nu\|u_{kx}\|\|u_{kxt}\| + \|f(u_k)\|\|u_{kxt}\| + \|g\|\|u_{kt}\|.\end{aligned}$$

By (2.10) and Agmon inequality

$$\|u\|_\infty \leqslant C\|u\|^{\frac{1}{2}}\|u\|^{\frac{1}{2}}_{H^1(\Omega)} \qquad \forall u \in H^1(\Omega) \tag{2.11}$$

we obtain that

$$\|f(u_k(t))\|_\infty \leqslant C \qquad \forall 0 \leqslant t \leqslant T. \tag{2.12}$$

And then it follows that

$$\|u_{kt}\|^2 + \|u_{kxt}\|^2 \leqslant C\|u_{kt}\| + C\|u_{kxt}\| \leqslant \tfrac{1}{2}\|u_{kt}\|^2 + \tfrac{1}{2}\|u_{kxt}\|^2 + C \tag{2.13}$$

which implies that

$$\|u_{kt}\|^2_{H^1(\Omega)} \leqslant 2C \qquad \forall 0 \leqslant t \leqslant T. \tag{2.14}$$

By (2.10) and (2.14) we find that there exist $\theta \in H^1(\Omega)$, $u(t) \in L^\infty(0, T; H^1(\Omega))$ and a subsequence, which is still denoted by u_k, such that

$$u_k(t_1) \to \theta \qquad \text{weakly in } H^1(\Omega) \tag{2.15}$$

$$u_k(t) \to u(t) \qquad \text{in } L^\infty(0, T; H^1(\Omega)) \text{ weak star} \tag{2.16}$$

$$u_{kt} \to u_t \qquad \text{in } L^\infty(0, T; H^1(\Omega)) \text{ weak star.} \tag{2.17}$$

By (2.16), (2.17) and a compactness theorem [16] we infer that

$$u_k(t) \to u(t) \qquad \text{in } L^2(0, T; H) \text{ strongly.} \tag{2.18}$$

$\forall v \in H^1_{\mathrm{per}}(\Omega)$, $\forall \psi(t) \in C_0^\infty(0, T)$, by (2.6) we claim that

$$\begin{aligned}&\int_0^T (u_{kt}, \psi(t)v)\,\mathrm{d}t + \int_0^T (u_{kxt}, \psi(t)v_x)\,\mathrm{d}t + \nu\int_0^T (u_{kx}, \psi(t)v_x)\,\mathrm{d}t \\ &\qquad -\int_0^T (f(u_k), \psi(t)v_x)\,\mathrm{d}t = \int_0^T (g, \psi(t)v)\,\mathrm{d}t.\end{aligned} \tag{2.19}$$

Note that

$$\left| \int_0^T (f(u_k), \psi(t)v_x)\,\mathrm{d}t - \int_0^T (f(u), \psi(t)v_x)\,\mathrm{d}t \right| = \left| \int_0^T (f(u_k) - f(u), \psi(t)v_x)\,\mathrm{d}t \right|$$
$$\leqslant \int_0^T \|f(u_k) - f(u)\| \|\psi(t)v_x\|\,\mathrm{d}t \leqslant \|f'(\xi)\|_\infty \int_0^T \|u_k - u\| \|\psi(t)v_x\|\,\mathrm{d}t$$
$$\leqslant C\|\psi(t)v_x\|_{L^2(0,T;H)} \|u_k - u\|_{L^2(0,T;H)} \to 0 \qquad \text{(by (2.18))}\ . \tag{2.20}$$

Taking the limit of (2.19) as $k \to \infty$, by (2.16), (2.17) and (2.20) we find that

$$\int_0^T (u_t, v)\psi(t)\,\mathrm{d}t + \int_0^T (u_{xt}, v_x)\psi(t)\,\mathrm{d}t + \nu \int_0^T (u_x, v_x)\psi(t)\,\mathrm{d}t$$
$$- \int_0^T (f(u), v_x)\psi(t)\,\mathrm{d}t = \int_0^T (g, v)\psi(t)\,\mathrm{d}t. \tag{2.21}$$

Hence, the following holds in the sense of distributions

$$u_t - u_{xxt} - \nu u_{xx} + (f(u))_x = g(x) \tag{2.22}$$

that is, $u(t)$ satisfies equation (2.1).

$\forall v \in H^1_{\mathrm{per}}(\Omega)$, $\forall \psi(t) \in C^\infty[0, T]$ with $\psi(T) = 0$, $\psi(0) = 1$, by (2.6) we obtain that

$$- \int_0^T (u_k, v)\psi'(t)\,\mathrm{d}t + \int_0^T (u_{kxt}, v_x)\psi(t)\,\mathrm{d}t + \nu \int_0^T (u_{kx}, v_x)\psi(t)\,\mathrm{d}t$$
$$- \int_0^T (f(u_k), v_x)\psi(t)\,\mathrm{d}t = (u_k(0), v) + \int_0^T (g, v)\psi(t)\,\mathrm{d}t. \tag{2.23}$$

Assumption (2.4) implies that

$$u_k(0) = u_{0k} \to w_0 \qquad \text{weakly in } H. \tag{2.24}$$

Then taking the limit of (2.23) as before, by (2.24) we obtain that

$$- \int_0^T (u, v)\psi'(t)\,\mathrm{d}t + \int_0^T (u_{xt}, v_x)\psi(t)\,\mathrm{d}t + \nu \int_0^T (u_x, v_x)\psi(t)\,\mathrm{d}t$$
$$- \int_0^T (f(u), v_x)\psi(t)\,\mathrm{d}t = (w_0, v) + \int_0^T (g, v)\psi(t)\,\mathrm{d}t. \tag{2.25}$$

On the other hand, by (2.22) we infer that

$$- \int_0^T (u, v)\psi'(t)\,\mathrm{d}t + \int_0^T (u_{xt}, v_x)\psi(t)\,\mathrm{d}t + \nu \int_0^T (u_x, v_x)\psi(t)\,\mathrm{d}t$$
$$- \int_0^T (f(u), v_x)\psi(t)\,\mathrm{d}t = (u(0), v) + \int_0^T (g, v)\psi(t)\,\mathrm{d}t. \tag{2.26}$$

It comes from (2.25), (2.26) that

$$(u(0), v) = (w_0, v) \qquad \forall v \in H^1_{\mathrm{per}}(\Omega).$$

This shows that

$$u(0) = w_0. \tag{2.27}$$

And thus by (2.22) and (2.27) we see that

$$u(t) = S(t)w_0 = w(t). \tag{2.28}$$

$\forall v \in H^1_{\rm per}(\Omega)$, $\forall \psi(t) \in C^\infty[0, t_1]$ with $\psi(0) = 0$, $\psi(t_1) = 1$, then repeating the procedure of proofs of (2.23)–(2.26), by (2.15) we find that

$$(u(t_1), v) = (\theta, v) \qquad \forall v \in H^1_{\rm per}(\Omega). \tag{2.29}$$

It follows from (2.28), (2.29) that

$$\theta = u(t_1) = S(t_1)w_0.$$

And then (2.15) implies that

$$S(t_1)u_{0k} \to S(t_1)w_0 \qquad \text{weakly in } H^1(\Omega)$$

which concludes theorem 2.1. □

3. The global attractor

In this section, we construct a global attractor for the problem (2.1)–(2.3). To the end, we assume that

$$g(x) \in L^2_{\rm per}(\Omega) \qquad \text{and} \qquad \int_\Omega g(x)\,{\rm d}x = 0. \tag{3.1}$$

And then integrating (2.1) over Ω and applying (2.3) we find that the average of $u(t)$ is conserved, i.e. for all $t > 0$:

$$\theta(u(t)) = \frac{1}{|\Omega|}\int_\Omega u(x,t)\,{\rm d}x = \frac{1}{|\Omega|}\int_\Omega u_0(x)\,{\rm d}x = \theta(u_0). \tag{3.2}$$

This shows that problem (2.1)–(2.3) has not bounded absorbing sets in the whole space H. This difficulty is overcome by introducing

$$H_\alpha = \{u \in H : |\theta(u)| \leqslant \alpha\}.$$

Equation (3.2) implies that H_α is invariant under the semigroup $S(t)$ associated to system (2.1)–(2.3).

In the sequel, we will show that bounded absorbing sets in H_α do indeed exist.

Lemma 3.1. Assume that (3.1) holds, $u_0 \in H^1_{\rm per}(\Omega) \bigcap H_\alpha$. Then for the solution $u(t)$ of problem (2.1)–(2.3) we have

$$\|u(t)\|_{H^1} \leqslant K \qquad \forall t \geqslant t_1$$

where K is a constant depending only on the data $(\nu, f, g, \Omega, \alpha)$, t_1 depends on the data $(\nu, f, g, \Omega, \alpha)$ and R when $\|u_0\|_{H^1} \leqslant R$.

In the following, we agree that $\forall u_0 \in H$,

$$\bar{u} = u - \theta(u) \qquad \theta(u) = \frac{1}{|\Omega|}\int_\Omega u(x)\,{\rm d}x. \tag{3.3}$$

Proof. We note that

$$\begin{aligned} u(t) &= \bar{u}(t) + \theta(u(t)) \qquad \text{(by (3.3))} \\ &= \bar{u}(t) + \theta(u_0) \qquad \text{(by (3.2))}. \end{aligned} \tag{3.4}$$

Substituting (3.4) into (2.1) we find that

$$\bar{u}_t - \bar{u}_{xxt} - \nu\bar{u}_{xx} + (f(u))_x = g(x). \tag{3.5}$$

Taking the inner product in H of (3.5) with $\bar{u}$ we infer that

$$\frac{1}{2}\frac{\rm d}{{\rm d}t}\|\bar{u}\|^2 + \frac{1}{2}\frac{\rm d}{{\rm d}t}\|\bar{u}_x\|^2 + \nu\|\bar{u}_x\|^2 + \int_\Omega (f(u))_x\bar{u}\,{\rm d}x = (g, \bar{u}). \tag{3.6}$$

Let

$$F(s) = \int_0^s f(\tau)\,\mathrm{d}\tau$$

then we find that

$$\int_\Omega (f(u))_x \bar{u}\,\mathrm{d}x = -\int_\Omega f(u)\bar{u}_x\,\mathrm{d}x = -\int_\Omega f(u)u_x\,\mathrm{d}x$$
$$= -\int_\Omega (F(u))_x\,\mathrm{d}x = -F(u(L)) + F(u(0)) = 0. \tag{3.7}$$

We recall the Poincaré inequality

$$\|v\| \leqslant C_1\|v_x\| \qquad \text{if } \int_\Omega v(x)\,\mathrm{d}x = 0. \tag{3.8}$$

By (3.3) we see that

$$\int_\Omega \bar{u}(x,t)\,\mathrm{d}x = \int_\Omega (u(x,t) - \theta(u))\,\mathrm{d}x$$
$$= \int_\Omega u(x,t)\,\mathrm{d}x - \int_\Omega u(x,t)\,\mathrm{d}x = 0 \qquad \text{(by (3.2))} \tag{3.9}$$

and then it comes from (3.8), (3.9) that

$$\|\bar{u}(t)\| \leqslant C_1\|\bar{u}_x(t)\| \qquad \forall t \geqslant 0. \tag{3.10}$$

Thus

$$|(g,\bar{u})| \leqslant \|g\|\|\bar{u}\| \leqslant C_1\|g\|\|\bar{u}_x\| \leqslant \tfrac{1}{4}\nu\|\bar{u}_x\|^2 + C_2 \qquad \text{(by Young inequality).} \tag{3.11}$$

In the sequel, we denote by C and C_i ($i = 1, 2, \ldots$) any constants depending only on the data $(\nu, f, g, \Omega, \alpha)$.

By (3.6), (3.7) and (3.11) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\bar{u}\|^2 + \|\bar{u}_x\|^2) + \frac{3}{2}\nu\|\bar{u}_x\|^2 \leqslant 2C_2. \tag{3.12}$$

Due to

$$\begin{aligned}\tfrac{3}{2}\nu\|\bar{u}_x\|^2 &= \tfrac{1}{2}\nu\|\bar{u}_x\|^2 + \nu\|\bar{u}_x\|^2\\ &\geqslant \tfrac{1}{2}\nu\|\bar{u}_x\|^2 + \nu C_1^{-2}\|\bar{u}\|^2 \qquad \text{(by (3.10))}\\ &\geqslant C_3(\|\bar{u}\|^2 + \|\bar{u}_x\|^2)\end{aligned} \tag{3.13}$$

where $C_3 = \min\{\frac{1}{2}\nu, \nu C_1^{-2}\}$.

By (3.12), (3.13) we claim that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\bar{u}\|^2 + \|\bar{u}_x\|^2) + C_3(\|\bar{u}\|^2 + \|\bar{u}_x\|^2) \leqslant 2C_2 \qquad \forall t \geqslant 0.$$

By the Gronwall lemma we obtain that

$$\begin{aligned}\|\bar{u}(t)\|^2 + \|\bar{u}_x(t)\|^2 &\leqslant (\|\bar{u}(0)\|^2 + \|\bar{u}_x(0)\|^2)\mathrm{e}^{-C_3 t} + \frac{2C_2}{C_3}\\ &\leqslant (1 + C_1^2)\|\bar{u}_x(0)\|^2\mathrm{e}^{-C_3 t} + \frac{2C_2}{C_3} \qquad \text{(by (3.10))}\\ &= (1 + C_1^2)\|u_x(0)\|^2\mathrm{e}^{-C_3 t} + \frac{2C_2}{C_3} \qquad \text{(by (3.3))}\\ &\leqslant (1 + C_1^2)R^2\mathrm{e}^{-C_3 t} + \frac{2C_2}{C_3} \qquad \forall t \geqslant 0\\ &\leqslant \frac{4C_2}{C_3} \qquad \forall t \geqslant t_*\end{aligned} \tag{3.14}$$

where $t_* = \frac{1}{C_3} \ln \frac{C_3(1+C_1^2)R^2}{2C_2}$.

Since

$$\int_\Omega \bar{u}(x,t)\theta(u(x,t))\,\mathrm{d}x = \theta(u(x,t))\int_\Omega \bar{u}(x,t)\,\mathrm{d}x = 0 \qquad \text{(by (3.9))} \tag{3.15}$$

and therefore we see that $\bar{u}(t)$ and $\theta(u(t))$ are orthogonal in H. Thus we obtain that

$$\begin{aligned}\|u(t)\|^2 &= \|\bar{u}(t)\|^2 + \|\theta(u)\|^2 = \|\bar{u}(t)\|^2 + |\theta(u)|^2|\Omega| \\ &\leqslant \|\bar{u}(t)\|^2 + \alpha^2|\Omega| \qquad \text{(by (3.2) and } u_0 \in H_\alpha).\end{aligned} \tag{3.16}$$

We claim that

$$\begin{aligned}\|u(t)\|_{H^1}^2 &= \|u(t)\|^2 + \|u_x(t)\|^2 \leqslant \|\bar{u}(t)\|^2 + \|\bar{u}_x(t)\|^2 + \alpha^2|\Omega| \qquad \text{(by (3.16))} \\ &\leqslant \frac{4C_2}{C_3} + \alpha^2|\Omega| \qquad \text{(by (3.14))}\end{aligned}$$

which concludes lemma 3.1. □

Obviously, lemma 3.1 implies that the ball

$$B = \{u \in H^1(\Omega) : \|u\|_{H^1} \leqslant K\} \tag{3.17}$$

is an absorbing set for $S(t)$ in $H^1(\Omega) \bigcap H_\alpha$.

Let

$$\mathcal{A} = \bigcap_{s\geqslant 0} \overline{\bigcup_{t\geqslant s} S(t)B} \tag{3.18}$$

where the closure is taken with respect to the H^1-weak topology. And then by theorem 2.1 we know that $\mathcal{A}$ is a global (weak) attractor for $S(t)$. More precisely, we have the following.

Theorem 3.1. Let (3.1) hold. Then the set $\mathcal{A}$ defined by (3.18) satisfies that

(i) $\mathcal{A}$ is bounded and weakly closed in $H^1(\Omega) \cap H_\alpha$;

(ii) $S(t)\mathcal{A} = \mathcal{A}$, $\forall t \geqslant 0$;

(iii) for every bounded set X in $H^1(\Omega)$, $S(t)X$ converges to $\mathcal{A}$ with respect to the H^1-weak topology as $t \to \infty$.

Proof. This theorem can be derived by theorem 2.1 and the techniques of [13]. The details are similar to that of [13], and so they are omitted here. □

We now apply the method introduced by Ball [15] to show that $\mathcal{A}$ is actually the global strong attractor in $H^1(\Omega)$.

Theorem 3.2. The global weak attractor $\mathcal{A}$ constructed in theorem 3 is actually the global strong attractor in $H^1(\Omega)$.

Proof. Since a point w belongs to the attractor $\mathcal{A}$ if and only if there exist two sequences $\{w_k^0\}_{k\in N}$ and $\{t_k\}_{k\in N}$ such that $S(t_k)w_k^0$ converges to w weakly in $H^1(\Omega)$, where $\{w_k^0\} \in B$, the bounded absorbing set in $H^1(\Omega)$, $t_k \to +\infty$. This theorem will be proved if we are able to show that for $w \in \mathcal{A}$, the sequence $S(t_k)w_k^0$ converges to w strongly in $H^1(\Omega)$.

Fix $T > 0$, by lemma 3.1 we know that the sequence $S(t_k - T)w_k^0$ is bounded in $H^1(\Omega)$, and then $v \in H^1(\Omega)$ and a subsequence exist, which are still denoted by $S(t_k - T)w_k^0$, such that

$$S(t_k - T)w_k^0 \to v \qquad \text{weakly in } H^1(\Omega). \tag{3.19}$$

Set $w_k(t) = S(t)S(t_k - T)w_k^0 = S(t_k + t - T)w_k^0$, by theorem 2.1 and (3.19) we see that $w_k(t) \to S(t)v$ weakly in $H^1(\Omega)$, and $w = S(T)v$.

We note that if $u(t)$ is a solution of problem (2.1)–(2.3), then by taking the inner product of (2.1) with u in H, we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|^2 + \|u_x\|^2) + 2\nu\|u_x\|^2 + 2\int_\Omega (f(u))_x u \,\mathrm{d}x = 2(g, u). \tag{3.20}$$

Similar to (3.7) we see that

$$2\int_\Omega (f(u))_x u \,\mathrm{d}x = -2\int_\Omega f(u)u_x \,\mathrm{d}x = 0. \tag{3.21}$$

And then by (3.20), (3.21) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{H^1} + 2\nu\|u(t)\|^2_{H^1} = 2\nu\|u(t)\|^2 + 2(g, u). \tag{3.22}$$

This shows that

$$\|u(t)\|^2_{H^1} = \mathrm{e}^{-2\nu t}\|u(0)\|^2_{H^1} + \int_0^t \mathrm{e}^{2\nu(\tau - t)} K(u(\tau))\,\mathrm{d}\tau \tag{3.23}$$

where

$$K(u) = 2\nu\|u\|^2 + 2(g, u). \tag{3.24}$$

By compactness embedding $H^1(\Omega) \subset H$ we can easily deduce that $K(u)$ is weakly continuous in $H^1(\Omega)$. Because any solution satisfies (3.23), and then for $w_k(t) = S(t)S(t_k - T)w_k^0$, we infer that

$$\|w_k(t)\|^2_{H^1} = \mathrm{e}^{-2\nu t}\|S(t_k - T)w_k^0\|^2_{H^1} + \int_0^t \mathrm{e}^{2\nu(\tau - t)} K(S(t_k + \tau - T)w_k^0)\,\mathrm{d}\tau. \tag{3.25}$$

Taking $t = T$ in (3.25), by the Lebesgue dominated convergence theorem we deduce that

$$\limsup_{k\to\infty}\|S(t_k)w_k^0\|^2_{H^1} \leqslant C\mathrm{e}^{-2\nu T} + \int_0^T \mathrm{e}^{2\nu(\tau - T)} K(S(\tau)v)\,\mathrm{d}\tau. \tag{3.26}$$

The above is obtained by the boundedness of $S(t_k - T)w_k^0$ in $H^1(\Omega)$.

Applying (3.23) for $w = S(T)v$ we have

$$\|w\|^2_{H^1} = \mathrm{e}^{-2\nu T}\|v\|^2_{H^1} + \int_0^T \mathrm{e}^{2\nu(\tau - T)} K(S(\tau)v)\,\mathrm{d}\tau. \tag{3.27}$$

By (3.26) and (3.27) we have

$$\limsup_{k\to\infty}\|S(t_k)w_k^0\|^2_{H^1} \leqslant C\mathrm{e}^{-2\nu T} + \|w\|^2_{H^1} - \mathrm{e}^{-2\nu T}\|v\|^2_{H^1}. \tag{3.28}$$

Let $T \to +\infty$, we find that

$$\limsup_{k\to\infty}\|S(t_k)w_k^0\|^2_{H^1} \leqslant \|w\|^2_{H^1}. \tag{3.29}$$

On the other hand, by the weak convergence of $S(t_k)w_k^0$ to w in $H^1(\Omega)$, we see that

$$\liminf_{k\to\infty}\|S(t_k)w_k^0\|^2_{H^1} \geqslant \|w\|^2_{H^1}. \tag{3.30}$$

And then we have

$$\lim_{k\to\infty}\|S(t_k)w_k^0\|^2_{H^1} = \|w\|^2_{H^1}. \tag{3.31}$$

This along with the weak convergence implies the strong convergence. The proof of theorem 3.2 is complete. □

Quite analogous to [14] where we dealt with the finite-dimensionality of the global attractor in $H^2(\Omega)$, we can also deduce the finite-dimensionality of the global attractor $\mathcal{A}$ here. That is, we have the following.

Theorem 3.3. The global attractor $\mathcal{A}$ of theorem 3.1 has finite fractal and Hausdorff dimensions in $H^1(\Omega)$.

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